From hyperbolic regularization to exact hydrodynamics for linearized Grad's equations

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Inspired by a recent hyperbolic regularization of Burnett's hydrodynamic equations [A. Bobylev, J. Stat. Phys. **124**, 371 (2006)], we introduce a method to derive hyperbolic equations of linear hydrodynamics to any desired accuracy in Knudsen number. The approach is based on a dynamic invariance principle which derives exact constitutive relations for the stress tensor and heat flux, and a transformation which renders the exact equations of hydrodynamics hyperbolic and stable. The method is described in detail for a simple kinetic model—a 13 moment Grad system.

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I. INTRODUCTION

Derivation of hydrodynamics from a microscopic description is the classical problem of physical kinetics. The Chapman-Enskog (CE) method [1] derives the solution from the Boltzmann equation in a form of a series in powers of Knudsen number ε, where ε is a ratio between the mean free path of a particle and the scale of variations of hydrodynamic fields. The Chapman-Enskog solution leads to a formal expansion of stress tensor and of heat flux vector in balance equations for density, momentum, and energy. Retaining the first order term (ε) in the latter expansions, we come to the Navier-Stokes equations, while next-order corrections are known as the Burnett [2] (ε^2) and the super-Burnett (ε^3) corrections [1]. It has long been conjectured that the inclusion of higher-order terms in the constitutive relations for the stress and heat flux may improve the predictive capabilities of hydrodynamics formulations in the continuum–transition regime where Navier-Stokes equations fail.

However, as it was first demonstrated by Bobylev for Maxwell's molecules [3], even in the simplest case (one-dimensional linear deviation from global equilibrium), the Burnett and the super-Burnett hydrodynamics violate the basic physics behind the Boltzmann equation. Namely, sufficiently short acoustic waves are increasing with time instead of decaying. Bobylev's instability has been also studied by Uribe *et al.* [4] for hard sphere molecules. This instability contradicts the *H* theorem, since all near-equilibrium perturbations must decay. This creates difficulties for an extension of hydrodynamics, as derived from a microscopic description, into a highly nonequilibrium domain where the Navier-Stokes approximation is inapplicable. For example, higher-order systems of hydrodynamic equations afforded a better description in certain situations such as shock structures on coarse grids, but are prone to small wavelength instabilities when grids are refined. Successes and drawbacks of the Burnett computations and a family of extended Burnett equations aimed at achieving entropy-consistent behavior of the equations have been recently summarized in [5].

The failure of the CE expansion does not lie in the method itself, but in its truncation to lower order levels. This question was studied in some detail for a class of simple kinetic models—Grad's moment systems [6]—in Refs. [7–12]. In these works, the Chapman-Enskog expansion was summed up exactly which revealed stability of exact hydrodynamics in contrast to its finite-order approximations. Alternative ways of approximating the Chapman-Enskog solution have been also suggested.

Very recently, Bobylev suggested a different viewpoint on the problem of Burnett's hydrodynamics [13]. Namely, violation of hyperbolicity can be seen as a source of instability. We remind that Boltzmann's and Grad's equations are hyperbolic and stable due to corresponding *H* theorems. However, the Burnett hydrodynamics is not hyperbolic which leads to no *H* theorem. Bobylev [13] suggested to stipulate hyperbolization of Burnett's equations which can also be considered as a change of variables. In this way hyperbolically regularized Burnett's equations admit the *H* theorem (in the linear case, at least) and stability is restored.

The aim of this paper is to study the issue of hyperbolicity of higher-order hydrodynamics in the case where the Chapman-Enskog solution can be found exactly. As a starting point, we consider the Grad's moment system, linearized at the equilibrium, and assuming unidirectional flow conditions (the 1D13M system, according to [7]). While simple enough, this model is nontrivial for three reasons: (i) application of the Chapman-Enskog method leads to a rather involved nonlinear recurrent relations for the coefficients of the expansion; (ii) the Burnett approximation derived from the Grad's moment system is identical to the one derived from the Boltzmann equation for Maxwell's molecules and thus violates hyperbolicity and exhibits Bobylev's instability [3]; (iii) even though the exact hydrodynamics can be derived following the lines of Refs. [7–12], and is stable, the question remains whether or not this exact hydrodynamics is manifestly hyperbolic.

The paper is organized as follows: In Sec. II, we derive exact hydrodynamics from the linearized 1D13M Grad's system. Derivation closely follows [12], and is based on application of a dynamic invariance principle which is equivalent to exact summation of the Chapman-Enskog expansion. A critical value of the Knudsen number is found beyond which a closed system of equations for the locally conserved fields ceases to exist. In Sec. III we find a class of transformations through which exact equations of hydrodynamics can be put in a hyperbolic form, thereby answering in affirmative the above question. We also analyze how such transformations affect the dissipative nature of the equations. In Sec. IV, we analyze and compare with conventional and earlier approximate solutions provided by (i) the Newton iteration method (Appendix A) and (ii) Bobylev's hyperbolic regularization of Burnett's equations (Appendix B) which turns out to be a special case of a more general result presented here. Finally, conclusions are provided in Sec. V.

II. HYDRODYNAMICS FROM THE LINEARIZED GRAD SYSTEM

A. Chapman-Enskog method and Bobylev's instability of Burnett's hydrodynamics

Point of departure is the linearized Grad's 13-moment system in one spatial variable x:

$$\partial_t\rho = -\partial_x u,$$

$$\partial_t u = -\partial_x\rho - \partial_x T - \partial_x\sigma,$$

$$\partial_t T = -\frac{2}{3}\partial_x u - \frac{2}{3}\partial_x q,$$

$$\partial_t\sigma = -\frac{4}{3}\partial_x u - \frac{8}{15}\partial_x q - \frac{1}{\varepsilon}\sigma,$$

$$\partial_t q = -\frac{5}{2}\partial_x T - \partial_x\sigma - \frac{2}{3\varepsilon}q. \tag{1}$$

Here $\rho(x,t)$, $u(x,t)$, and $T(x,t)$ are the reduced deviations of density, average velocity, and temperature from their equilibrium values, respectively, and $\sigma(x,t)$ and $q(x,t)$ are the reduced xx component of the nonequilibrium stress tensor and heat flux, respectively. Moreover, $\varepsilon>0$ has a meaning of the Knudsen number. The latter is given by the ratio between the mean free path λ and a characteristic dimension of the system L and is the smallness parameter in the Chapman-Enskog method [1]. The magnitude of the Knudsen number determines the appropriate gas dynamic regime [14]. In a sequel, we use rescaled variables $t'=\varepsilon t$ and $x'=\varepsilon x$ and omit the prime to simplify notation.

The system (1) provides the time evolution equations for a set of hydrodynamic (locally conserved) fields $[\rho,u,T]$ coupled to the nonhydrodynamic fields σ and q. The goal is to reduce the number of equations in Eq. (1) and to arrive at a closed system of three equations for the hydrodynamic fields only. Thanks to linearity of the system (1) it proves convenient to turn into the reciprocal space, and seek for solutions of the form $\zeta=\zeta_k\exp(\omega t+ikx)$, where ζ is a generic function ρ,u,T,σ,q, and where k is a real valued wave number.

Application of the Chapman-Enskog (CE) method to the reduction of the system (1) results in the following series expansion of the nonhydrodynamic variables into the powers of k:

$$\sigma_k = \sum_{n=0}^{\infty}\sigma_k^{(n)}, \quad q_k = \sum_{n=0}^{\infty}q_k^{(n)}, \tag{2}$$

where the coefficients $\sigma_k^{(n)}$ and $q_k^{(n)}$ are of order $\sigma_k^{(n)}\sim k^{n+1}$, $q_k^{(n)}\sim k^{n+1}$, and are obtained from a recurrence procedure:

$$\sigma_k^{(n)} = -\left\{\sum_{m=0}^{n-1}\partial_t^{(m)}\sigma_k^{(n-1-m)} + \frac{8}{15}ikq_k^{(n-1)}\right\},$$

$$q_k^{(n)} = -\left\{\sum_{m=0}^{n-1}\partial_t^{(m)}q_k^{(n-1-m)} + ik\sigma_k^{(n-1)}\right\}, \tag{3}$$

and where the CE operators $\partial_t^{(m)}$ act on the hydrodynamic fields as follows:

$$\partial_t^{(m)}\rho_k = \begin{cases}-iku_k, & m=0\\ 0, & m\geq 1\end{cases},$$

$$\partial_t^{(m)}u_k = \begin{cases}-ik(\rho_k+T_k), & m=0\\ -ik\sigma_k^{(m-1)}, & m\geq 1\end{cases},$$

$$\partial_t^{(m)}T_k = \begin{cases}-\frac{2}{3}iku_k, & m=0\\ -\frac{2}{3}ikq_k^{(m-1)}, & m\geq 1\end{cases}. \tag{4}$$

It can be proven that functions σ_k and q_k have the following structure, for all $n=0,1,\ldots$:

$$\sigma_k^{(2n)} = a_n(-k^2)^n iku_{k,},$$

$$\sigma_k^{(2n+1)} = b_n(-k^2)^{n+1}\rho_k + c_n(-k^2)^{n+1}T_k,$$

$$q_k^{(2n)} = x_n(-k^2)^n ik\rho_k + y_n(-k^2)^n ikT_k,$$

$$q_k^{(2n+1)} = z_n(-k^2)^{n+1}u_k, \tag{5}$$

where $a_n,\ldots,z_n$ are numerical coefficients to be determined. Note the altering structure of expansion coefficients of odd and even orders. Substituting Eq. (5) into Eqs. (3) and (4), the CE method casts into recurrence equations in terms of the coefficients $a_n,\ldots,z_n$:

$$a_{n+1} = b_n + \frac{2}{3}c_n + \frac{2}{3}\sum_{m=1}^{n}c_{n-m}z_{m-1} + \sum_{m=0}^{n}a_{n-m}a_m - \frac{8}{15}z_n,$$

$$b_{n+1} = a_{n+1} + \sum_{m=0}^{n}a_{n-m}b_m + \frac{2}{3}\sum_{m=0}^{n}c_{n-m}x_m - \frac{8}{15}x_{n+1},$$

$$c_{n+1} = a_{n+1} + \sum_{m=0}^{n}a_{n-m}c_m + \frac{2}{3}\sum_{m=0}^{n}c_{n-m}y_m - \frac{8}{15}y_{n+1},$$

$$x_{n+1} = z_n + \sum_{m=1}^{n}z_{n-m}b_{m-1} + \frac{2}{3}\sum_{m=0}^{n}y_{n-m}x_m - b_n,$$

$$y_{n+1} = z_n + \sum_{m=1}^{n} z_{n-m}c_{m-1} + \frac{2}{3}\sum_{m=0}^{n} y_{n-m}y_m - c_n,$$

$$z_{n+1} = x_{n+1} + \frac{2}{3}y_{n+1} + \frac{2}{3}\sum_{m=0}^{n} y_{n-m}z_m + \sum_{m=0}^{n} z_{n-m}a_m - a_{n+1}. \tag{6}$$

System (6) is solved recurrently subject to the initial conditions,

$$a_0 = -\frac{4}{3}, \quad b_0 = -\frac{4}{3}, \quad c_0 = \frac{2}{3}, \quad x_0 = 0, \quad y_0 = -\frac{15}{4},$$

$$z_0 = -\frac{7}{4}. \tag{7}$$

The initial conditions are obtained by evaluating the functions σ_k and q_k up to the Burnett order [see Eq. (15) below] and identifying the coefficients a_0, x_0, and y_0 from the Navier-Stokes approximation and the remaining coefficients b_0, c_0, and z_0 from the Burnett correction. Equation (6) defines six functions,

$$A(k) = \sum_{n=0}^{\infty} a_n(-k^2)^n, \ldots, Z(k) = \sum_{n=0}^{\infty} z_n(-k^2)^n. \tag{8}$$

Thus the CE solution amounts to finding functions $A,\ldots,Z$ [Eq. (8)]. Note that by the nature of the CE recurrence procedure, functions $A,\ldots,Z$ [Eq. (8)] are real-valued functions. Knowing $A,\ldots,Z$ [Eq. (8)], we can express the nonequilibrium stress tensor and heat flux as

$$\sigma_k = ikA(k)u_k - k^2B(k)\rho_k - k^2C(k)T_k, \tag{9}$$

$$q_k = ikX(k)\rho_k + ikY(k)T_k - k^2Z(k)u_k. \tag{10}$$

Upon substituting these expressions into the Fourier-transformed balance equations (1), we obtain the closed system of hydrodynamic equations which is conveniently written in a vector form,

$$\partial_t \mathbf{x} = \mathbf{M}\mathbf{x}, \tag{11}$$

where $\mathbf{x} \equiv (\rho_k, u_k, T_k)$, and the matrix $\mathbf{M}$ has the form

$$\mathbf{M} = \begin{pmatrix} 0 & -ik & 0 \\ -ik(1-k^2B) & k^2A & -ik(1-k^2C) \\ \frac{2}{3}k^2X & -\frac{2}{3}ik(1-k^2Z) & \frac{2}{3}k^2Y \end{pmatrix}. \tag{12}$$

With this, we find the dispersion relation for the hydrodynamic modes $\omega(k)$ by solving the characteristic equation,

$$\det(\mathbf{M} - \omega\mathbf{I}) = 0, \tag{13}$$

with $\mathbf{I}$ the unit matrix.

The standard application of the CE procedure is to approximate functions $A,\ldots,Z$ by polynomials with coefficients found from the recurrence procedure (6). The first nonvanishing contribution is the Newton-Fourier constitutive relations,

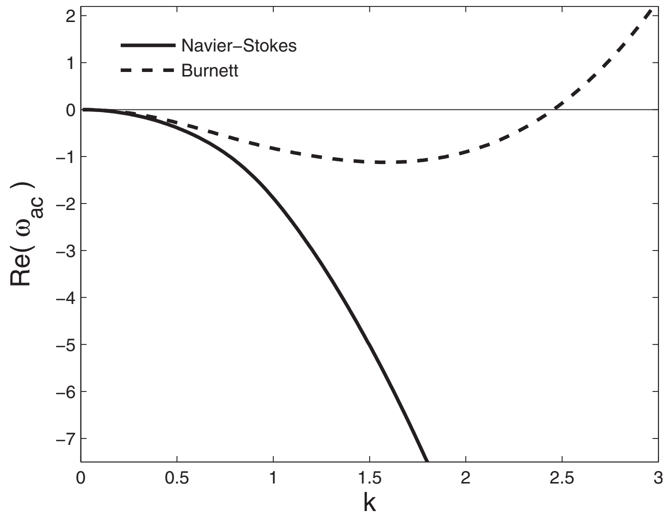

FIG. 1. Dispersion relation. Acoustic mode $\mathrm{Re}(\omega_{ac})$ for Navier-Stokes and Burnett hydrodynamics.

$$\sigma_k^{(0)} = -\frac{4}{3}iku_k, \quad q_k^{(0)} = -\frac{15}{4}ikT_k. \tag{14}$$

which leads to the Navier-Stokes-Fourier hydrodynamic equations. Computing the coefficients $\sigma_k^{(1)}$ and $q_k^{(1)}$, we arrive at the Burnett level:

$$\sigma_k = -\frac{4}{3}iku_k + \frac{4}{3}k^2\rho_k - \frac{2}{3}k^2T_k,$$

$$q_k = -\frac{15}{4}ikT_k + \frac{7}{4}k^2u_k. \tag{15}$$

The Burnett approximation (15) coincides with that obtained by Bobylev [3] from the Boltzmann equation for Maxwell's molecules. Unlike the Navier-Stokes-Fourier approximation, the Burnett constitutive relations (15) show instability of the acoustic mode, see Fig. 1.

Thus the difficulty of the CE method consists in the way the functions $A,\ldots,Z$ are approximated, the standard polynomial approximations lead to unstable hydrodynamic equations. We shall now derive closed-form equations for these functions which amounts to summing up the CE series exactly.

B. Invariance equations

Summation of the CE series for the functions $A,\ldots,Z$ can be done directly from the recurrence relations (6) following the lines of Ref. [8]. Alternatively but completely equivalently, one can make use of the dynamic invariance principle (DIP) [7]. Here, the set of nonhydrodynamic moments $\{\sigma, q\}$ is still thought in the form (9) and (10), but the method makes no assumption about the power-series representation of the functions $A,\ldots,Z$. The time derivative of $\{\sigma, q\}$ can be computed in two different ways. On the one hand, substituting Eqs. (9) and (10) into the moment system (1), we find

$$\partial_t \sigma = -\frac{4}{3} i k u_k - \frac{8}{15} i k q(X,Y,Z,k) - \sigma(A,B,C,k),$$

$$\partial_t q = -\frac{5}{2} i k T_k - i k \sigma(A,B,C,k) - \frac{2}{3} q(X,Y,Z,k).$$

On the other hand, the time derivative of $\{\sigma, q\}$ can be computed due to the closed hydrodynamic equations by chain rule:

$$\partial_t \sigma = \frac{\partial \sigma}{\partial u_k}\partial_t u_k + \frac{\partial \sigma}{\partial \rho_k}\partial_t \rho_k + \frac{\partial \sigma}{\partial T_k}\partial_t T_k, \qquad (16a)$$

$$\partial_t q = \frac{\partial q}{\partial u_k}\partial_t u_k + \frac{\partial q}{\partial \rho_k}\partial_t \rho_k + \frac{\partial q}{\partial T_k}\partial_t T_k. \qquad (16b)$$

Here, the derivatives $\partial_t u_k$ and $\partial_t T_k$ are evaluated self-consistently using the functions (9) and (10) in the right hand side of Eq. (1). The DIP states that the two time derivatives coincide, since the set $\{\sigma, q\}$ has to solve both the full Grad system and the reduced system. This requirement implies a closed set of equations, here referred as invariance equations (IE), relating the six functions $A(k), \ldots, Z(k)$:

$$-\frac{4}{3} - A - k^2\left(A^2 + B - \frac{8Z}{15} + \frac{2C}{3}\right) + \frac{2}{3}k^4 CZ = 0,$$

$$\frac{8}{15}X + B - A + k^2 AB + \frac{2}{3}k^2 CX = 0,$$

$$\frac{8}{15}Y + C - A + k^2 AC + \frac{2}{3}k^2 CY = 0,$$

$$A + \frac{2}{3}Z + k^2 ZA - X - \frac{2}{3}Y + \frac{2}{3}k^2 YZ = 0,$$

$$k^2 B - \frac{2}{3}X - k^2 Z + k^4 ZB - \frac{2}{3}k^2 YX = 0,$$

$$-\frac{5}{2} - \frac{2}{3}Y + k^2(C - Z) + k^4 ZC - \frac{2}{3}k^2 Y^2 = 0. \qquad (17)$$

The same equations can be derived upon summation of the CE expansion. Equations (17) are a convenient starting point for evaluation of exact hydrodynamics. For $k=0$ one recovers the initial conditions (7).

C. Exact hydrodynamic solutions

The dispersion relation $\omega(k)$ was found by simultaneously solving numerically the invariance equations (17) and the characteristic equation (13). The resulting hydrodynamic spectrum consist of two modes, the acoustic mode $\omega_{\text{ac}}(k)$, represented by two complex-conjugated roots of Eq. (13), and the real-valued diffusive heat mode $\omega_{\text{diff}}(k)$, cf. Fig. 2.

Among the many sets of solutions $\{A(k), \ldots, Z(k)\}$ to the system (17), the relevant ones are continuous functions with

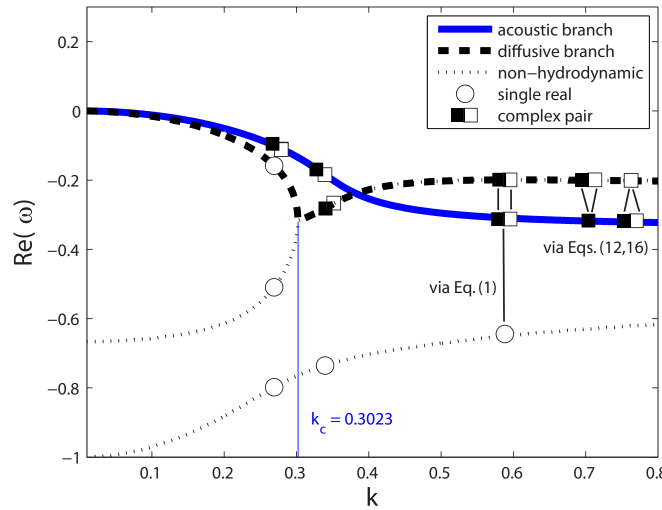

FIG. 2. (Color online) Dispersion relation for the linearized 1D13M Grad system (1). The unique solution of hydrodynamical modes obtained from Eq. (13) with Eq. (17) coincides with the real parts of the modes of the original Grad system (the plot also shows when pairs of conjugate complex roots appear), the solution of the original system (1) features five ω's, while the exact solution of Eq. (13) with Eq. (17) has three ω's for each k and degenerated over the hydrodynamic branches at $k \geq k_c$.

the asymptotics: $\lim_{k\to 0}\omega_{\text{hydr}}=0$. Remarkably, we find that the solution with this asymptotics is unique, and represented by a pair of complex conjugated sets, $[\mathcal{S}, \mathcal{S}^*]$, shown in Figs. 3 and 4. Note that a qualitative change of dynamics arises when the diffusive mode couples with one of the two nonhydrodynamical modes of Grad's system at a critical wave number $k_c \approx 0.3023$, which is the value where also the Newton method diverges, cf. Appendix A. By the CE perspective, the hydrodynamics of the diffusive mode stops at k_c, since, after that point, it becomes a complex-valued function coupled with the conjugated nonhydrodynamic mode, see

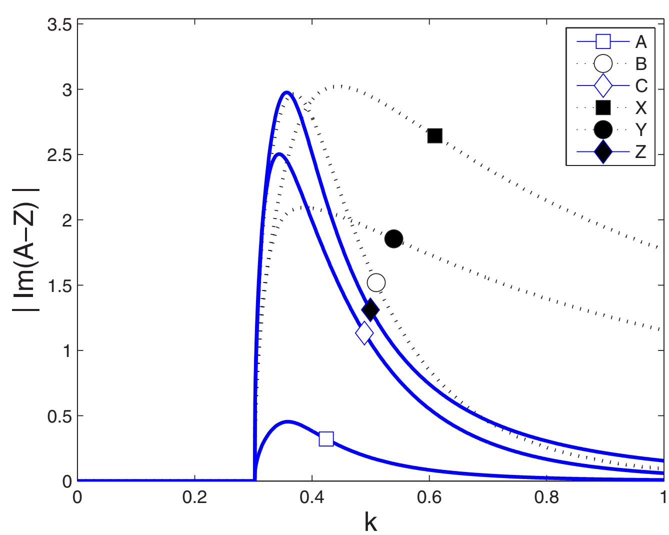

FIG. 3. (Color online) Imaginary parts of coefficients A to Z solving Eq. (17). Shown is the unique solution leading to hydrodynamic branches, cf. Fig. 2, which is symmetric about the real axis.

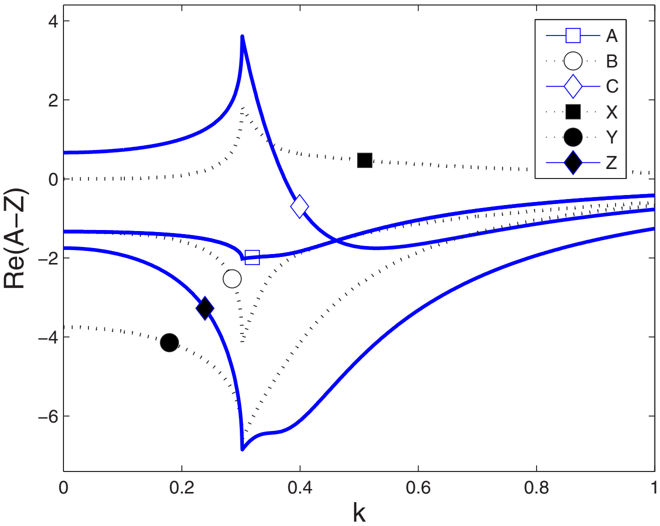

FIG. 4. (Color online) Real parts of complex-valued functions $A,\ldots,Z$ solving Eq. (17). It is clearly visible that the solution matches the initial condition (7).

Fig. 2. Essentially, for $k \geq k_c$, the CE method does not recognize any longer the resulting diffusive branch as an extension of a hydrodynamic branch. Also, the set of solutions $[\mathcal{S},\mathcal{S}^*]$, real valued for $k \leq k_c$, continues upon a complex manifold, cf. Fig. 3. We notice that the occurrence of a pair of complex conjugated sets of solution is very plausible due to symmetry reasons: inserting $\mathcal{S}$ into the dispersion relation, we obtain a pair of complex conjugated acoustic modes $[\omega_{\mathrm{ac}}(\mathcal{S},k),\omega_{\mathrm{ac}}^*(\mathcal{S},k)]$ plus one of the complex modes resulting from the extension of the diffusive branch for $k \geq k_c$; whereas, through $\mathcal{S}^*$, we obtain, symmetrically, the two latter conjugated modes, plus one of the conjugated acoustic modes.

As a further evidence of this close coupling, we also notice the occurrence of an intersection between the real parts of the hydrodynamical modes $\mathrm{Re}(\omega_{\mathrm{ac}})$ and $\mathrm{Re}(\omega_{\mathrm{diff}})$ after the critical point, at $k=k_{\mathrm{coupl}} \approx 0.383$. Therefore the message extracted from the study of Grad's system (1) is that the set of hydrodynamic equations for $[\rho,u,T]$ provides, as expected, stable solutions, when taking into account all the orders of CE expansion—which corresponds to solving the system of invariance equations (17). And, there is no closed set of hydrodynamic equations after k_c, even though the acoustic mode extends smoothly beyond k_c, as is visible in Fig. 2.

Thus the exact hydrodynamics as derived by the summation of the CE expansion (or, equivalently, from the invariance equations) extends up to a finite critical value k_c. No stability violation occurs, unlike in the finite-order truncations thereof. While we have evaluated the functions $A,\ldots,Z$ numerically, two questions remained open: (i) Is the (stable) exact hydrodynamics also hyperbolic? (ii) If so, how to retain hyperbolicity in the approximations? In the next section we shall answer the first of these questions.

III. HYPERBOLIC TRANSFORMATION FOR EXACT HYDRODYNAMICS

Equation (1) for the Fourier component vector $\mathbf{x} \equiv (\rho_k,u_k,T_k)$ reads $\partial_t\mathbf{x}=\mathbf{M}\mathbf{x}$ with $\mathbf{M}$ from Eq. (12). By explicitly re-introducing the Knudsen number ε, i.e., by replacing k by $k\varepsilon$ in $\mathbf{M}$ and further distinguishing between the real and imaginary matrix elements in $\mathbf{M}$, we can write

$$\partial_t\mathbf{x} = [\mathrm{Re}(\mathbf{M}) - i\,\mathrm{Im}(\mathbf{M})]\mathbf{x},$$

$$\mathrm{Re}(\mathbf{M}) = \sum_{n=0}^{\infty}(-1)^n\mathbf{R}^{(n)}\varepsilon^{2n+1} = \varepsilon\mathbf{R}^{(0)} - \varepsilon^3\mathbf{R}^{(1)} + O(\varepsilon^5), \tag{18}$$

$$\mathrm{Im}(\mathbf{M}) = \sum_{n=0}^{\infty}(-1)^n\mathbf{I}^{(n)}\varepsilon^{2n} = \mathbf{I}^{(0)} - \varepsilon^2\mathbf{I}^{(1)} + \varepsilon^4\mathbf{I}^{(2)} - O(\varepsilon^6), \tag{19}$$

rearranged such that the Knudsen number expansion coefficients become visible. We find that the operators $\mathrm{Re}(\mathbf{M})$ (real part) and $\mathrm{Im}(\mathbf{M})$ (imaginary part) involve the following real-valued operators (for all $n \geq 0$, i.e., with the convention $a_{-1}=c_{-1}=z_{-1}\equiv 1$ and Kronecker symbol δ),

$$\mathbf{I}^{(n)} = k^{2n+1}\begin{pmatrix} 0 & \delta_{n,0} & 0 \\ b_{n-1} & 0 & c_{n-1} \\ 0 & \frac{2}{3}z_{n-1} & 0 \end{pmatrix},$$

$$\mathbf{R}^{(n)} = k^{2n+2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_n & 0 \\ \frac{2}{3}x_n & 0 & \frac{2}{3}y_n \end{pmatrix}. \tag{20}$$

Equations of hydrodynamics (18) are hyperbolic and stable provided that we can find a transformation of hydrodynamic fields such that (i) $\mathrm{Re}(\mathbf{M})$ and $\mathrm{Im}(\mathbf{M})$ are both real and symmetric, and (ii) $\mathrm{Re}(\mathbf{M})$ has negative semidefinite eigenvalues. Therefore we seek a transformation $\mathbf{z}=\mathbf{T}\mathbf{x}$ which produces a symmetric matrix $\mathbf{M}'=\mathbf{T}\mathbf{M}\mathbf{T}^{-1}$ and we wish to see if $\mathrm{Re}(\mathbf{M}')=\mathrm{Re}(\mathbf{T}\mathbf{M}\mathbf{T}^{-1})$ is negative semidefinite. We consider the equations of exact hydrodynamics, i.e., Eqs. (18) provided that functions $A,\ldots,Z$ [Eq. (20)] are solutions to the invariance equations (17). After a few algebra which we do not recapitulate here, we obtain a particular transformation matrix $\mathbf{T}$ which solves the problem. It is a member of a whole class of effectively equivalent transformations, and can be written as

$$\mathbf{T} = \frac{1}{T_{uu}}\begin{pmatrix} T_{\rho\rho} & 0 & T_{\rho T} \\ 0 & T_{uu} & 0 \\ 0 & 0 & T_{TT} \end{pmatrix}, \tag{21}$$

with the nonvanishing components

$$T_{\rho\rho} = \frac{T_{uu}^2}{\sqrt{3X+2Y[[Z]]}},$$

$$T_{uu} = \sqrt{X[[3B-2Z[[C]]-2C]]+2Y[[B]][[Z]]},$$

$$T_{\rho T} = -\frac{3[[C]]X}{\sqrt{3X+2Y[[Z]]}},$$

$$T_{TT} = \sqrt{3[[C]](Y[[B]]-[[C]]X)}, \tag{22}$$

where we have introduced the following symbolic notation:

$$[[\bullet]] \equiv 1-(k\varepsilon)^2 \bullet . \tag{23}$$

The transformation $\mathbf{T}$ [Eq. (21)] symmetrizes $\mathbf{M}$ and renders the system hyperbolic, as can be verified by straightforward computation of $\mathbf{M}'$ from Eqs. (21), (25b). We further notice that $\mathbf{T}$ contains only even powers of $(k\varepsilon)$ because the same is true for the coefficients A–Z.

Next we calculate the eigenvalues $\lambda_{1,2,3}$ of $\mathrm{Re}(\mathbf{M}')$—containing transport coefficients—to obtain a remarkably simple result:

$$\lambda_1 = 0, \quad \lambda_2 = k^2\varepsilon A, \quad \lambda_3 = \frac{2}{3}k^2\varepsilon Y. \tag{24}$$

From the analysis of the previous section, it follows that the nontrivial eigenvalues (24) are negative semidefinite for all $k\varepsilon$ (see Fig. 4 which displays the exact numerical solutions for A and Y). Hence the equation describing hyperbolic hydrodynamics (also hyperbolic up to an arbitrarily selected order ε^n, a feature to be used in the next section) attains the form

$$\partial_t \mathbf{z} = \mathbf{M}'\mathbf{z}, \tag{25a}$$

$$\mathbf{M}' = \mathbf{T}\mathbf{M}\mathbf{T}^{-1} \tag{25b}$$

for the vector $\mathbf{z}=\{\tilde{\rho}_k, \tilde{u}_k, \tilde{T}_k\}=\mathbf{T}\mathbf{x}$ of transformed hydrodynamic variables, and where $\mathbf{M}'$ is symmetric and has seminegative eigenvalues. To summarize,

$$\text{hyperbolicity: } (\mathbf{M}')^T = \mathbf{M}', \tag{26a}$$

$$\text{dissipativity: } \begin{cases} \mathrm{Tr}[\mathrm{Re}(\mathbf{M}')] \le 0, \\ \det[\mathrm{Re}(\mathbf{M}')] \ge 0. \end{cases} \tag{26b}$$

Equation (25) with Eqs. (21) and (12) satisfying Eq. (26) is the main result of this paper. The occurrence of negative eigenvalues in the exact solutions, together with the existence of a transformation $\mathbf{T}$ which makes the equations hyperbolic, proves that exact hydrodynamics (1), without approximations, is stable. In the remainder of this paper we shall make use of the hyperbolicity of exact hydrodynamics in order to establish approximate hydrodynamic equations which retain this property.

IV. LOWER ORDER HYPERBOLIC AND STABLE HYDRODYNAMICS

A. Approximations on the hyperbolic equations

In applications, one is interested in using truncated hydrodynamic equations by taking into account only lower orders of the Knudsen number ε. In this case, the functions $A,\ldots,Z$ are replaced by their lower-order approximations, and they can be generally written—as shown already in Eq. (8)—as polynomials truncated to an arbitrary order n. Their coefficients are usually derived through the CE recurrence equations, as outlined above. With the exact numerical solution at hand, we can also find, at any given order of approximation, the optimal interpolating functions $A,\ldots,Z$ solving Eq. (17), a method we wish to recommend, and which has been worked out in Table I. Exact hydrodynamics, as described by Grad's system (1), terminates at k_c. In this regime we can perform a Taylor expansion, up to any order n, upon the elements of all the three matrices $\mathbf{T}$, $\mathbf{M}$, and $\mathbf{T}^{-1}$. Thus the approximations are done on the manifestly hyperbolic equation (25) in such a way as to retain hyperbolicity and stability in each order of approximation. It is worthwhile noticing that the eigenvalues, upon approximating Eq. (25) to a polynomial order n, transform in a canonical manner:

$$\lambda_1^{(n)} = 0, \quad \lambda_2^{(n)} = k^2\varepsilon\left(a_0 + \sum_{m=1}^{n} a_m (k\varepsilon)^m\right),$$

$$\lambda_3^{(n)} = \frac{2}{3}k^2\varepsilon\left(y_0 + \sum_{m=1}^{n} y_m (k\varepsilon)^m\right), \tag{27}$$

and, depending upon the polynomial coefficients, and in particular depending on the sign of the highest order coefficients a_n, y_n, the eigenvalues $\lambda_{2,3}$ diverge to $\pm\infty$ for $k\varepsilon\to\infty$, but stay negative for $k\le k_c$, if we use coefficients according to the method summarized in Table I. We shall now consider a few examples of the suggested procedure.

B. Euler and Navier-Stokes equations

For the zeroth-order term, $\mathrm{Im}(\mathbf{M})=\mathbf{I}^{(0)}$ (Euler), the transformation is, according to Eq. (22), given by a diagonal matrix with entries $T_{\rho\rho}=T_{uu}=1$ and $T_{TT}=\sqrt{3/2}$, all eigenvalues are identically zero. The first order term, linear in the Knudsen number (Navier-Stokes) is obviously stable as well; all eigenvalues are negative semidefinite since $a_0=-4/3$ and $y_0=-15/4$ are both negative.

C. Hyperbolic regularization for the Burnett level

The Burnett equations are unstable without regularization. For this level of description, second order in the Knudsen number ε, with $\mathrm{Im}(\mathbf{M})=\mathbf{I}^{(0)}-\varepsilon^2\mathbf{I}^{(1)}$, upon inserting the required exact solutions at vanishing wave number, $a_0,\ldots,z_0$ from Eq. (17), cf. Table I, into Eqs. (21) and (22), the transformation matrix achieving a symmetric $\mathrm{Im}(\mathbf{M}')$ reads

$$\mathbf{T} = \begin{pmatrix} 1+\frac{2}{3}(k\varepsilon)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}-\frac{29}{8\sqrt{6}}(k\varepsilon)^2} \end{pmatrix}. \tag{28}$$

This transformation coincides with the one derived by Bobylev's hyperbolic regularization method [13], specified for the present model (an alternate derivation which follows

TABLE I. Polynomial coefficients introduced in Eq. (8) obtained from the exact numerical solution, cf. Fig. 4, by requiring that deviations between exact and polynomial series at a given order of the method (first column) stay below 1% (i.e., would be invisible in the plot). We used the symmetrized functions $A(k)+A(-k)$ over the whole real axes for k when performing the fits in order to enforce correct symmetry. This criterion corresponds to a regularization procedure which produces stable results up to the limit $(k\varepsilon)\leq(k\varepsilon)_c=0.3023$, as is easily verified, and leads to a recommended range of (high precision) applicability of the method (second column). For convenience we list faculty-rescaled series coefficients. These coefficients are essentially the prefactors for higher order correction terms in hydrodynamic equations and can be used to study the intermediate Knudsen number regime $0\ll k\varepsilon<(k\varepsilon)_c$. As described in the text, with a suitable transformation matrix $\mathbf{T}$ these choices lead to very convenient hyperbolic differential equations for the hydrodynamic fields $\mathbf{x}=(\rho,u,T)$.

Method	apply at	n	$a_n/n!$	$b_n/n!$	$c_n/n!$	$x_n/n!$	$y_n/n!$	$z_n/n!$
0	$k\varepsilon\leq0.03$	0	−4/3	−4/3	2/3	0	−15/4	−7/4
1	$k\varepsilon\leq0.17$	0	−4/3	−4/3	2/3	0	−15/4	−7/4
		1	1.132	2.536	−3.735	−0.716	5.873	9.953
2	$k\varepsilon\leq0.25$	0	−4/3	−4/3	2/3	0	−15/4	−7/4
		1	0.706	1.156	−2.500	0.309	4.652	7.053
		2	−1.304	−4.095	3.720	3.030	−3.741	−8.902
3	$k\varepsilon\leq0.28$	0	−4/3	−4/3	2/3	0	−15/4	−7/4
		1	1.104	3.042	−4.055	−1.123	5.903	9.718
		2	0.329	3.669	−2.678	−2.861	1.398	2.023
		3	0.648	3.083	−2.540	−2.340	2.040	4.333

closely Ref. [13] is given in Appendix B). Notice that up to the Burnett level only the polynomial coefficients at vanishing wave number, listed in the first row of Table I, enter the transformation $\mathbf{T}$, which can be indirectly also inferred from the eigenvalues, cf. Eq. (27).

D. Beyond the Burnett level

In Table I, we provide not only coefficients, but also ranges of applicability for the given coefficients of $A-Z$ which can be used if we extend the procedure to higher order. The optimal coefficients are provided by the least squares fit of the numerical data for exact hydrodynamics, see Table I. Within the given ranges, the eigenvalues of $\mathrm{Re}(\mathbf{M}')$ are negative semidefinite, i.e., the spectrum of the acoustic mode $\omega_{\mathrm{ac}}(k)$ of the corresponding hyperbolic hydrodynamic system is then stable for all wavelengths.

E. Application: Hyperbolic regularization for the super-Burnett level

Finally, in order to present explicit illustration of the approximation strategy presented in Sec. III, we present the equations on the next, super-Burnett, level, which takes into account terms up to the order $(k\varepsilon)^3$. The equations of change for the transformed variables $\mathbf{z}$ read

$$\partial_t\mathbf{z}=\mathbf{M}'\mathbf{z}, \tag{29}$$

with a symmetric $\mathbf{M}'$,

$$\mathbf{M}'=-ik\left\{\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & \sqrt{\frac{2}{3}}\\ 0 & \sqrt{\frac{2}{3}} & 0\end{pmatrix}+(k\varepsilon)^2\begin{pmatrix}0 & \frac{2(5+x_1)}{15} & 0\\ \frac{2(5+x_1)}{15} & 0 & \frac{65-24x_1}{60\sqrt{6}}\\ 0 & \frac{65-24x_1}{60\sqrt{6}} & 0\end{pmatrix}\right\}-k^2\varepsilon\left\{\begin{pmatrix}0 & 0 & 0\\ 0 & \frac{4}{3} & 0\\ 0 & 0 & \frac{5}{2}\end{pmatrix}+(k\varepsilon)^2\begin{pmatrix}0 & 0 & \sqrt{\frac{2}{3}}x_1\\ 0 & a_1 & 0\\ \sqrt{\frac{2}{3}}x_1 & 0 & \frac{2}{3}y_1\end{pmatrix}\right\}, \tag{30}$$

and transformation

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} \end{pmatrix} + (k\varepsilon)^2 \begin{pmatrix} \frac{2(5-x_1)}{15} & 0 & \frac{2x_1}{5} \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{145+24x_1}{40\sqrt{6}} \end{pmatrix}, \tag{31}$$

where third order terms are not present because $\mathbf{T}$ is symmetric in k. The Burnett level (28), where x_1 disappears, is immediately recovered from Eq. (31). The hydrodynamic variables are restored using $\mathbf{x}=\mathbf{T}^{-1}\mathbf{z}$, with the inverse transformation (suitable at the super-Burnett level), which, due to our (arbitrarly) chosen normalization factor T_{uu} in Eq. (21) only slightly differs from $\mathbf{T}$:

$$\mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix} - (k\varepsilon)^2 \begin{pmatrix} \frac{2(5-x_1)}{15} & 0 & \frac{2x_1}{5}\sqrt{\frac{2}{3}} \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{145+24x_1}{60\sqrt{6}} \end{pmatrix}. \tag{32}$$

To complete the "simulation algorithm" using Eqs. (29)–(32), we need numerical values for x_1 and y_1, and an initial condition for $\mathbf{x}$, or $\mathbf{z}$. One solves the hyperbolically stable system for $\mathbf{z}$, and finally calculate $\mathbf{x}$ via $\mathbf{T}^{-1}$. Suitable values for the coefficients are those given in Table I for method 1: $y_1=5.873$ and $x_1=-0.716$, because higher order coefficients such as x_2 do not enter. The equations of this section should allow us to study the regime $0 \le k\varepsilon < 0.17$ very accurately. For the remaining regime, $0.17 < k\varepsilon < (k\varepsilon)_c$, the presented equations are also stable and hyperbolic, but not as accurate. They are, by definition, more accurate compared with the ones obtained using the recursion method. The equations offered in this section serve as an example on how to use our more general result, Eq. (21).

V. CONCLUSIONS

In this paper, we have considered derivation of hydrodynamics for a simple model (1) for which—as we have demonstrated—all details can be explicitly studied. The main finding is that the exact hydrodynamic equations (summation of the Chapman-Enskog expansion to all orders) are manifestly hyperbolic and stable. To the best of our knowledge, this is the first complete answer of the kind. We have also suggested a way of approximating the higher order hydrodynamic equations using accurate numerical solution of the invariance equations and expansion of the transformation which renders the system hyperbolic. The study supports the recent suggestion of Bobylev on the hyperbolic regularization of Burnett's approximation, and reduces to the latter in a special case.

We conclude this paper with a few comments on the possible extensions of the present approach. (i) The technique of deriving exact hydrodynamics and/or hyperbolic approximations thereof can be readily applied to linearized Grad's systems with a larger number of moments. In particular, we were able to extend the present derivation to the three-dimensional 13 moment system, the results qualitatively agree with the one-dimensional case considered above and will be reported separately. (ii) It is also possible to apply the present techniques to derive exact hydrodynamics from the dynamically corrected Grad's systems, first introduced in [15] and studied in some detail in [16]. The latter equations have arguably better properties than the Grad's equations, especially in the moderate Knudsen number regime where the linearized systems become relevant to study of microflows. (iii) In this paper, we were addresing the boundary conditions for neither the Grad's systems nor for the higher order hydrodynamic equations. As is well known, this question remains essentially open for both. Therefore a different and intriguing field of applications of the present technique is the recently established lattice Boltzmann hierarchy (LBH) [17–25]. Although the primitive variables in the LBH are populations of carefully chosen discrete velocities, the LBH equations can be cast into a form of moment systems similar to Grad's equations. The crucial advantage of the LBH above Grad's systems is that the former is equipped with pertinent boundary conditions derived directly from the Maxwell-Boltzmann theory [19]. Recently, it has been demonstrated, both numerically and analytically, that the LBH is capable of capturing such phenomena as slip and nonlinear Knudsen layers [24,25]. The present techniques can be applied for reducing higher order lattice Boltzmann models with advantage for the numerics. However, this goes beyond the scope of this paper, the interested reader is directed to [23–25] for details.

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APPENDIX A: NEWTON ITERATION

The analytical complexity of either the CE method or the invariance equations is overwhelming when we regard systems other than the linearized Grad system, such as the Boltzmann equation. Approximate solutions are, then, the only feasible approach. In this section we shall describe the appli-

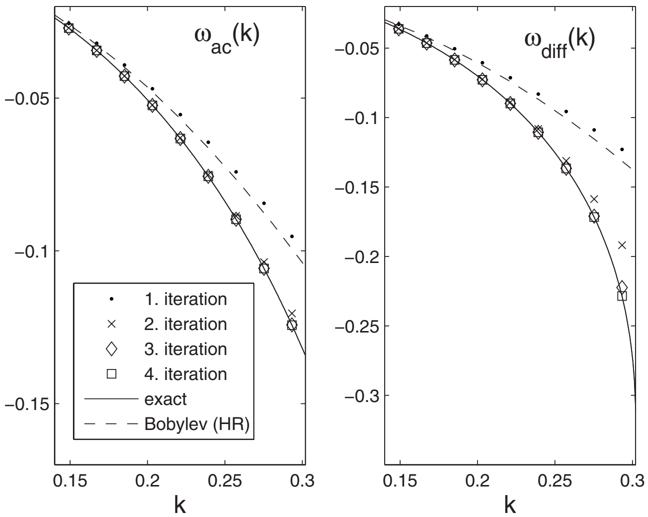

FIG. 5. Dispersion relations $\omega(k)$ for acoustic and diffusive modes obtained via Newton iteration. In the plots, shown is also the approximation obtained through Bobylev's hyperbolic regularization (HR) [13]. Newton iterations fail for $k \geq k_c = 0.3023$.

cation of the Newton iteration method to the invariance equations. We used Newton's method, cf. Fig. 5, to solve iteratively Eqs. (1), taking, as the initial condition, the Euler approximation (corresponding to a nondissipative hydrodynamics: $A_0 = \cdots = Z_0 = 0$), which leads, after the first iteration, to the same result achievable, alternatively, through a technique of partial summation [7] of the CE expansion: essentially, a sort of regularized Burnett approximation. It is seen in Fig. 5 that Newton iterations converge rapidly to the exact hydrodynamics in the domain of its validity, $k \leq k_c$.

APPENDIX B: BOBYLEV'S HYPERBOLIC REGULARIZATION

This appendix reviews a recent approach by Bobylev [13] and establishes a connection to the second-order variant of our approach. We use the original notation of Ref. [13] to facilitate comparisons.

As was demonstrated in Ref. [13], after truncating the CE expansion at the Burnett level, the (linearized) equation of hydrodynamics takes the general form $\partial_t x + i(B_0 + \varepsilon^2 B_1)x + \varepsilon A x + O(\varepsilon^3) = 0$, where x is the vector of hydrodynamics variables $[\rho, u, T]$ and the operators B_0, A, and B_1 refer, respectively, to the Euler, Navier-Stokes, and Burnett level of approximation. $B = B_0 + \varepsilon^2 B_1$ is a real nonsymmetric operator for $\varepsilon > 0$. When applied to the Grad's system (1), these findings are a special cases of Eq. (19) with Eq. (20) upon identifying $B_n = (-1)^n \mathbf{I}^{(n)}$, $A = A_0$, and $A_n = (-1)^n \mathbf{R}^{(n)}$. The loss of symmetry of the operator B was identified as the reason of the instability occurring in the Burnett equations. In order to cure this loss of symmetry, HR introduces a symmetric real valued operator$R$ and defines a change of variables such that $z = x + \varepsilon^2 R x$—or in our notation above, $\mathbf{T} = (\mathbf{1} + \varepsilon^2 R)$. Hence the resulting equation of hydrodynamics attains the form $z_t + i[B_0 + \varepsilon^2(B_1 + RB_0 - B_0 R)]z + \varepsilon A z + O(\varepsilon^3) = 0$, more generally $\mathbf{z} = \mathbf{TMT}^{-1}\mathbf{z}$. The suggested regularization consists in writing $\mathbf{T}^{-1}$ as a polynomial (Taylor) expansion in powers of ε and in truncating it, as for $\mathbf{T}$, at second order.

HR therefore provides a regularization which is exact up to the order ε^2. The operator R has to be chosen in such a way that $\tilde{B}_1 = B_1 + [R, B_0]$ is real and symmetric, where $[R, B_0] \equiv RB_0 - B_0 R$. It is instructive to consider the HR as applied to the example of the 1D13M (1) which has originally been written, in matrix notation, as

$$\partial_t \begin{pmatrix} \rho_k \\ u_k \\ T_k \end{pmatrix} = \left\{ -i \left[\begin{pmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & \frac{2}{3}k & 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 & 0 & 0 \\ \frac{4}{3}k^3 & 0 & -\frac{2}{3}k^3 \\ 0 & \frac{7}{6}k^3 & 0 \end{pmatrix} \right] - \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{3}k^2 & 0 \\ 0 & 0 & \frac{5}{2}k^2 \end{pmatrix} \right\} \begin{pmatrix} \rho_k \\ u_k \\ T_k \end{pmatrix} + O(\varepsilon^3). \tag{B1}$$

Expression (B1) offers those first terms of Eq. (19) with Eq. (20) for which the coefficients $(a_0 = -4/3, \ldots, z_0 = -7/4)$ are analytically known, cf. Table I for all values. Equation (B1) can hence equivalently be formulated as $\mathbf{M} = \varepsilon \mathbf{R}^{(0)} - i(\mathbf{I}^{(0)} - \varepsilon^2 \mathbf{I}^{(1)}) + O(\varepsilon^3)$. To apply the regularization procedure to the system (1), one needs to make matrix B_0 symmetric (it corresponds to restoring the hyperbolicity of Euler equations, through a transformation $\mathbf{T}_\alpha$). Then, introducing a real-valued, symmetric (diagonal) matrix R with diagonal elements $a(k)$, $b(k)$, and $c(k)$ (which corresponds choosing a diagonal $\mathbf{T}$), and imposing the symmetry of the resulting operator $\tilde{B}_1$ (more generally, of $\mathbf{TI}^{(n)}\mathbf{T}^{-1}$), the coefficients are interrelated as follows [13]:

$$a(k) = b(k) + \frac{2}{3}k^2, \quad c(k) = b(k) - \frac{29}{24}k^2. \tag{B2}$$

Notice the transformation $R_0 R$ is a special case of Eq. (21). The resulting operator $\tilde{B}_1$ is given by

$$\tilde{B}_1 = B_1 + [R, B_0] = B_1 + b(k)[I, B_0] + [m_{ij}, B_0] = B_1 + [m_{ij}, B_0], \tag{B3}$$

and therefore unique [independent of $b(k)$]. Hence the hydrodynamic equations resulting from HR as applied to 1D13M attain the form

$$\partial_t \begin{pmatrix} \rho_k \\ u_k \\ T_k \end{pmatrix} = - \begin{pmatrix} 0 & ik\left(1+\frac{2}{3}k^2\varepsilon^2\right) & 0 \\ ik\left(1+\frac{2}{3}k^2\varepsilon^2\right) & \frac{4}{3}k^2\varepsilon & \sqrt{\frac{2}{3}}ik\left(1+\frac{13}{24}k^2\varepsilon^2\right) \\ 0 & \sqrt{\frac{2}{3}}ik\left(1+\frac{13}{24}k^2\varepsilon^2\right) & \frac{5}{2}k^2\varepsilon \end{pmatrix} \begin{pmatrix} \rho_k \\ u_k \\ T_k \end{pmatrix} + O(\varepsilon^3). \tag{B4}$$

Since Eq. (B4) is a special case of the more general Eqs. (29) and (30), the connection to Bobylev's work has been explicitly established.

[1] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, New York, 1970).
[2] D. Burnett, Proc. London Math. Soc. **39**, 385 (1935).
[3] A. V. Bobylev, Sov. Phys. Dokl. **27**, 29 (1982).
[4] F. J. Uribe, R. M. Velasco, and L. S. Garcìa-Colìn, Phys. Rev. E **62**, 5835 (2000).
[5] R. Balakrishnan, J. Fluid Mech. **503**, 201 (2004).
[6] H. Grad, Commun. Pure Appl. Math. **2**, 331 (1949).
[7] A. N. Gorban and I. V. Karlin, *Invariant Manifolds for Physical and Chemical Kinetics*, Lecture Notes in Physics Vol. 660 (Springer, Berlin, 2005).
[8] A. N. Gorban and I. V. Karlin, Phys. Rev. Lett. **77**, 282 (1996).
[9] I. V. Karlin, G. Dukek, and T. F. Nonnenmacher, Phys. Rev. E **55**, 1573 (1997).
[10] I. V. Karlin, G. Dukek, and T. F. Nonnenmacher, Phys. Rev. B **55**, 6324 (1997).
[11] I. V. Karlin, J. Phys. A **33**, 8037 (2000).
[12] I. V. Karlin and A. N. Gorban, Ann. Phys. **11**, 783 (2002).
[13] A. V. Bobylev, J. Stat. Phys. **124**, 371 (2006).
[14] M. Kröger and M. Hütter, J. Chem. Phys. **125**, 044105 (2006).
[15] I. V. Karlin, A. N. Gorban, G. Dukek, and T. F. Nonnenmacher, Phys. Rev. E **57**, 1668 (1998).
[16] H. Struchtrup and M. Torrilhon, Phys. Fluids **15**, 2668 (2003).
[17] X. Shan and X. He, Phys. Rev. Lett. **80**, 65 (1998).
[18] I. V. Karlin, A. Ferrante, and H. C. Öttinger, Europhys. Lett. **47**, 182 (1999).
[19] S. Ansumali and I. V. Karlin, Phys. Rev. E **66**, 026311 (2002).
[20] S. Ansumali, I. V. Karlin, and H. C. Öttinger, Europhys. Lett. **63**, 798 (2003).
[21] S. Ansumali and I. V. Karlin, Phys. Rev. Lett. **95**, 260605 (2005).
[22] S. S. Chikatamarla, S. Ansumali, and I. V. Karlin, Phys. Rev. Lett. **97**, 010201 (2006).
[23] S. S. Chikatamarla and I. V. Karlin, Phys. Rev. Lett. **97**, 190601 (2006).
[24] S. Ansumali, I. V. Karlin, S. Arcidiacono, A. Abbas, and N. I. Prasianakis, Phys. Rev. Lett. **98**, 124502 (2007).
[25] I. V. Karlin and S. Ansumali, e-print arXiv:cond-mat/0703437.